W3/1 Injective and Flat Modules

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Cor 1.12 Let $(P_i)_{i \in I}$ be a family of modules. Then

$$\left(\forall i \in I : P_i \text{ projective} \right) \iff \bigoplus_{i \in I} P_i \text{ projective}$$

Proof: see Exercises (easy)

Thm 1.13 For $E \in A\text{-Mod}$, TFAE

(a) E is injective (i.e. $\mathrm{Hom}(-, E)$ is exact)

(b) For every mono $f : M \to N$, $f^* : \mathrm{Hom}(N, E) \to \mathrm{Hom}(M, E)$ is an epi.

(c) For every diagram with exact row (in black)

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\varphi"'] & N \arrow[dl, dashed, "\exists \psi"] \\ & E & \end{tikzcd}$$

$\exists \psi \in \mathrm{Hom}(N, E) : \quad \varphi = \psi \circ f$

(d) Every mono $f : E \to M$ splits (i.e. $\exists r : M \to E$ s.t. $r \circ f = \mathrm{id}_E$)

(e) [Baer's Criterion] For every $I \trianglelefteq A$, $j : I \hookrightarrow A$ the inclusion, and every diagram (black)

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook, "j"'] \arrow[d, "\varphi"'] & A \arrow[dl, dashed, "\exists \psi"] \\ & E & \end{tikzcd}$$

$\exists \psi \in \mathrm{Hom}(A, E) : \quad \varphi = \psi \circ j$.

Proof: (a) $\iff$ (b) ✓ Since $\mathrm{Hom}(-, E)$ is left exact in any case.

(b) $\iff$ (c): $\mathrm{Hom}(N, E) \xrightarrow{f^*} \mathrm{Hom}(M, E)$, $\alpha \mapsto \alpha \circ f$ is surjective

$\iff \forall \varphi \in \mathrm{Hom}(M, E) \; \exists \psi \in \mathrm{Hom}(N, E) : \quad \varphi = \psi \circ f$

$\Leftrightarrow \forall \varphi \in \mathrm{Hom}(M,E)\ \exists \psi \in \mathrm{Hom}(N,E): \quad \varphi = \psi \circ f$

(c) $\Rightarrow$ (d) Consider

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "f"] \arrow[d, "id_E"'] & M \arrow[dl, dashed, "\exists r"] \\ & E & \end{tikzcd}$$

By (c), $\exists r: M \to E: \quad r \circ f = id_E$

(d) $\Rightarrow$ (c) Consider

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\varphi"'] & N \arrow[d, "j_N"] \\ & E \arrow[r, shift right, "j_E"'] & Q \arrow[l, shift right, "r"'] \end{tikzcd} \qquad \text{(exact row)}$$

Let $K := \{(\varphi(m), -f(m)) \in E \times N \ : \ m \in M\}$ and $Q := E \times N / K$

(pushout of f and φ)

There are hom's $j_E: E \to Q$, $e \mapsto (e,0)+K$, $j_N: N \to Q$, $n \mapsto (0,n)+K$,

j_E is a mono: $j_E(e) = 0 \Rightarrow (e,0) \in K \Rightarrow \exists m \in M: \ e = \varphi(m)$ and $0 = f(m)$

$\overset{f \text{ mono}}{\Longrightarrow} m = 0 \Rightarrow e = \varphi(m) = 0.$

$\overset{(d)}{\Longrightarrow} \exists r: Q \to E: \quad r \circ j_E = id_E$

Define $\psi: N \to E$ as $\psi = r \circ j_N$.

$\Rightarrow \forall m \in M: \quad \varphi(m) = r \circ j_E \circ \varphi(m) = r((\varphi(m), 0) + K) \overset{\substack{-(\varphi(m), -f(m)) + K \\ \text{in argument}}}{=}$

$= r((0, f(m)) + K) = r \circ j_N \circ f(m) = \psi \circ f(m)$

(c) $\Rightarrow$ (e) ✓

(e) $\Rightarrow$ (c) Consider

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\varphi"', no head] & N \\ & {} & \end{tikzcd}$$

$$\begin{array}{c} \Big\downarrow \varphi \\ E \end{array}$$

Let $\Omega := \{ (N', \psi') : f(M) \leq N' \leq N, \ \psi' \in \mathrm{Hom}(N', E) : \psi' \circ f = \varphi \}$

There is a partial order on Ω:

$$(N', \psi') \leq (N'', \psi'') \iff N' \subseteq N'' \text{ and } \psi''|_{N'} = \psi'.$$

- $\Omega \neq \emptyset$ as $(f(M), \varphi \circ f^{-1}) \in \Omega$
- If $\Omega_0 \subseteq \Omega$ is a chain (meaning: $\forall (N', \psi'), (N'', \psi'') \in \Omega_0$,
  $(N', \psi') \leq (N'', \psi'')$ or $(N'', \psi'') \leq (N', \psi')$)

  then also $\Big(\underbrace{\bigcup_{N' \in \Omega_0} N'}_{\sum_{N' \in \Omega_0} N'}, \psi_0 \Big) \in \Omega$, where $\psi_0|_{N'} = \psi' \ \forall (N', \psi') \in \Omega_0$

  [easy details omitted]

$\xRightarrow{\text{Zorn's Lemma}}$ $\exists$ a maximal $(N', \psi') \in \Omega$.

Claim: $N' = N$.

Suppose $N' \subsetneq N$. Let $x \in N \setminus N'$. Then

$I := \{ a \in A : ax \in N' \}$ is an ideal of A.

Define $\mu_I : I \to E$, $a \mapsto \psi'(ax)$. By (e), μ_I extends to $\mu : A \to E$.

Let $N'' := N' + Ax$, $\psi'' : N'' \to E$, $\underset{N'}{\overset{\in}{n}} + \underset{A}{\overset{\in}{a}} x \mapsto \psi'(n) + \mu(a)$

[Well-defined: Suppose $n + ax = n' + a'x$ with $n, n' \in N'$, $a, a' \in A$

$\Rightarrow n - n' = (a' - a)x \Rightarrow a' - a \in I \Rightarrow \mu(a') - \mu(a) = \mu(\underbrace{a' - a}_{\in I}) =$

$= \psi'((a' - a)x) = \psi'(n - n') = \psi'(n) - \psi'(n')$

$\Rightarrow \psi'(n) + \mu(a) = \psi'(n') + \mu(a')$]

$\Rightarrow \psi'(n)+\mu(a) = \psi'(n')+\mu(a')$,

Then $(N'', \psi'') \in \Omega$, contradicting maximality of (N', ψ') $\square$

<u>Exm:</u> <u>$A=\mathbb{Z}$:</u> $\mathbb{Z}_{\mathbb{Z}}$ not injective:

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r, hook] \arrow[d] & \mathbb{Z} \\ & \mathbb{Z} & \end{tikzcd} \qquad (n \neq 0) \qquad n \mapsto 1$$

does not extend if $n \neq \pm 1$

$\mathbb{Q}_{\mathbb{Z}}$ is injective:

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r, hook] \arrow[d, "f"] & \mathbb{Z} \arrow[dl, dashed] \\ & \mathbb{Q} & \end{tikzcd} \qquad n \mapsto q, \quad 1 \mapsto q/n$$

<u>Def</u> Let A be a domain. $M \in A\text{-Mod}$ is divisible if

$$\forall m \in M \ \forall a \in A^{\bullet} \ \exists m_0 \in M: \quad m = a m_0$$

<u>E.g.</u> $\mathbb{Q}_{\mathbb{Z}}$, $(\mathbb{Q}/\mathbb{Z})_{\mathbb{Z}}$ are divisible, $(\mathbb{Z}/n\mathbb{Z})_{\mathbb{Z}}$, $n \neq \pm 1$ is not .

Following the previous example, one easily shows (Exercise):

<u>Prop 1.14</u> Let A be a PID, $M \in A\text{-Mod}$. Then

M injective $\Leftrightarrow$ M is divisible.

<u>Cor 1.15</u> Let $(E_i)_{i \in I}$ be a family of modules. Then

$(\forall i \in I: \ E_i$ is injective$) \Leftrightarrow \prod_{i \in I} E_i$ injective (Exercise)

<u>Lemma 1.16</u> (Snake lemma) Given a diagram (in A-Mod)

$$\begin{tikzcd} & M \arrow[r, "i"] \arrow[d, "f"] & N \arrow[r] \arrow[d, "g"] & P \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & N' \arrow[r, "p"] & P' & \end{tikzcd}$$

with exact rows, there is an exact sequence

$$\mathrm{Ker}(f) \xrightarrow{i|_{\mathrm{Ker}(f)}} \mathrm{Ker}(g) \to \mathrm{Ker}(h) \to \mathrm{Coker}(f) \to \mathrm{Coker}(g) \xrightarrow{\bar{p}} \mathrm{Coker}(h) .$$

connecting hom.

$\mathrm{Ker}(f) \xrightarrow{u|_{\mathrm{Ker}(f)}} \mathrm{Ker}(g) \to \mathrm{Ker}(h) \to \mathrm{Coker}(f) \to \mathrm{Coker}(g) \to \mathrm{Coker}(h)$.

↑ connecting hom.

If i is mono, so is $i|_{\mathrm{Ker}(f)}$

If p is epi, so is $\bar{p}$.

(Proof by diagram chasing, see Exercises)

<u>Lemma 1.17</u> Let $E \in \mathrm{Mod}\text{-}A$, and $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ a split SES. Then $0 \to E \otimes M \xrightarrow{E\otimes f} E \otimes N \xrightarrow{E \otimes g} E \otimes P \to 0$ is a split SES.

<u>Proof Sketch:</u> Apply $E \otimes -$ to

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{f} & N & \xrightarrow{g} & P & \to & 0 \\ & & \downarrow{\scriptstyle id_M} & & \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle id} & & \\ 0 & \to & M & \hookrightarrow & M \oplus P & \to & P & \to & 0 \end{array} \qquad \text{(cf. L 1.7)}$$

$m \mapsto (m, 0)$

$(0, p) \mapsto p$

and use that $E \otimes -$ preserves direct sums (i.e. the bottom row) $\square$

<u>Remark:</u> $\mathrm{Hom}(M, -)$, $\mathrm{Hom}(-, M)$ have the same property. More generally all additive functors have this property.

<u>Thm 1.18</u> For $E \in A\text{-Mod}$ TFAE:

(a) E is flat (i.e., $E \otimes_A -$ is exact)

(b) For every mono $f: M \to N$, also $E \otimes f: E \otimes M \to E \otimes N$, $e \otimes m \mapsto e \otimes f(m)$ is a mono.

(c) For every fin. gen. $I \trianglelefteq A$, the hom

$$\mu_I: I \otimes E \to E, \quad x \otimes e \mapsto xe$$

is a mono (hence, induces an iso $I \otimes E \cong IE$)

<u>Proof:</u> (a) $\Leftrightarrow$ (b) ✓ using right exactness of $E \otimes_A -$

(b) $\Rightarrow$ (c) Apply (b) to $I \hookrightarrow A$ and use $A \otimes E \cong E$, $a \otimes e \mapsto ae$.

(b) $\Rightarrow$ (c) Apply (b) to $I \hookrightarrow A$ and use $A \otimes E \cong E$, $a \otimes e \mapsto ae$.

(c) $\Rightarrow$ (b) Let $f: M \to N$ be a mono.

Observation 1: If $\varphi: N \to N'$ is an iso, it suffices to show the claim for the inclusion $j: \underbrace{\varphi(f(M))}_{=: M'} \hookrightarrow N' = \varphi(N)$.

[
$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \wr\downarrow \varphi\circ f & & \wr\downarrow \varphi \\ M' & \xhookrightarrow{j} & N' \end{array} \quad \text{yields} \quad \begin{array}{ccc} E\otimes M & \xrightarrow{E\otimes f} & E\otimes N \\ \wr\downarrow E\otimes(\varphi\circ f) & & \wr\downarrow E\otimes\varphi \\ E\otimes M' & \xrightarrow{E\otimes j} & E\otimes N', \end{array}$$

so $\operatorname{Ker}(E\otimes f) \cong \operatorname{Ker}(E\otimes j)$]

Observation 2: Let $j: M \hookrightarrow N$ an inclusion map. If $E \otimes j': E\otimes M' \to E \otimes N$ is injective for all f.g. $M' \leq M$, then $E\otimes j: E\otimes M \to E\otimes N$ is injective.

[Let $x := \sum_{i=1}^{s} e_i \otimes m_i \in \operatorname{Ker}(E\otimes j)$, with $e_i \in E$, $m_i \in M$.

Let $M' := \langle m_1, \dots, m_s\rangle_A \leq M$, and $j': M' \hookrightarrow N$, $\varepsilon: M' \hookrightarrow M$ the inclusions. Now the following diagram commutes

$$\begin{tikzcd} E\otimes M' \arrow[r, "E\otimes\varepsilon"] \arrow[rr, bend left, "E\otimes j'"] & E\otimes M \arrow[r, "E\otimes j"] & E\otimes N \\ x' \arrow[r, mapsto] & x \arrow[r, mapsto] & 0 \end{tikzcd}$$

Let $x' := \sum_{i=1}^{s} e_i \otimes m_i \in E\otimes M' \Rightarrow (E\otimes\varepsilon)(x') = x$

$$\Rightarrow 0 = (E\otimes j)\circ(E\otimes\varepsilon)(x') = (E\otimes j')(x') \overset{E\otimes j' \text{ mono}}{\Longrightarrow} x' = 0 \Rightarrow x = 0. \;]$$

Case 1: $M \subseteq N$, $j: M \hookrightarrow N$ inclusion, N f.g. free

Induction on $r := rk(N) \in \mathbb{N}_0$. $\underline{r=0}$: $M = N = 0$ ✓

$\underline{r=1}$: $\Rightarrow N \cong A$, claim holds by assumption & Obs. 1 & Obs 2.

$r-1 \to r$, $r \geq 2$: $\Rightarrow N = F_1 \oplus F_2$ with $rk(F_1) = r-1$, $rk(F_2) = 1$,

$r=1$: $N = A$, claim holds by assumption

$\underline{r-1 \to r,\ r \geq 2}$: $\Rightarrow N = F_1 \oplus F_2$ with $\operatorname{rk}(F_1) = r-1$, $\operatorname{rk}(F_2) = 1$,

$\pi_i : N \to F_i$ canonical epis, $\varepsilon_i : F_i \hookrightarrow N$ canonical embeddings.

Let $M_1 := F_1 \cap M = \varepsilon_1^{-1}(M)$, $M_2 := \pi_2(M) \cong M/M_1$

Then the following diagram has exact rows, w. bottom split exact.

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, hook, "\varepsilon_1|_{M_1}"] \arrow[d, "j_1"] & M \arrow[r, "\pi_2|_M"] \arrow[d, "j"] & M_2 \arrow[r] \arrow[d, "j_2"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "\varepsilon_1"] & N \arrow[r, "\pi_2"] & F_2 \arrow[r] & 0 \end{tikzcd}$$

($j_1 = j \mid M_1$, j_2 obtained by factoring $\pi_2 \circ j$ through M_1)

Tensor with E:

$$\begin{tikzcd} & \operatorname{Ker}(E \otimes j_1) \arrow[r] \arrow[d] & \operatorname{Ker}(E \otimes j) \arrow[r] \arrow[d] & \operatorname{Ker}(E \otimes j_2) \arrow[d] & \\ & E \otimes M_1 \arrow[r] \arrow[d, "E \otimes j_1"] & E \otimes M \arrow[r] \arrow[d, "E \otimes j"] & E \otimes M_2 \arrow[r] \arrow[d, "E \otimes j_2"] & 0 \\ 0 \arrow[r] & E \otimes F_1 \arrow[r] & E \otimes N \arrow[r] & E \otimes F_2 \arrow[r] & 0 \end{tikzcd}$$

(Snake lemma)

(exact rows & cols; exactness on bottom by split exactness!)

By IH, $\operatorname{Ker}(E \otimes j_1) = \underline{0} = \operatorname{Ker}(E \otimes j_2) \overset{\text{Snake L}}{\Longrightarrow} \operatorname{Ker}(E \otimes j) = 0$.

$\underline{\text{Case 2}}$: $j : M \hookrightarrow N$ inclusion, N free

Using Obs. 2 again, it suffices to show this for M f.g.

Let $N = \bigoplus_{i \in I} A e_i$ w. $(e_i)_{i \in I}$ an A-basis of N

M f.g. $\Rightarrow \exists I_0 \subseteq I$: I_0 finite, $M \subseteq N_0 := \bigoplus_{i \in I_0} A e_i$.

Let $\varepsilon_M : M \hookrightarrow N_0$, $\varepsilon_0 : N_0 \hookrightarrow N$ be the inclusions $\Rightarrow j = \varepsilon_0 \circ \varepsilon_M$

$\Rightarrow E \otimes j = (E \otimes \varepsilon_0) \circ (E \otimes \varepsilon_M)$ is a mono

$$\Longrightarrow \quad E\otimes j = \underset{\substack{\uparrow \\ \text{mono bec. } \varepsilon_0 \text{ split} \\ \text{mon (L1.17)}}}{(E\otimes \varepsilon_0)} \circ \underset{\substack{\uparrow \\ \text{mono by Case 1}}}{(E\otimes \varepsilon_M)} \quad \text{is a mono}$$

<u>Case 3:</u> $j: M \hookrightarrow N$ inclusion, N arbitrary.

Let $\pi: F \to N$ be an epi with F free, $K := \mathrm{Ker}(\pi)$

We have:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook] \arrow[d, "id_K"] & \pi^{-1}(M) \arrow[r, "\pi|_{\pi^{-1}(M)}"] \arrow[d] & M \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & K \arrow[r, hook] & F \arrow[r, "\pi"] & N \arrow[r] & 0,
\end{tikzcd}$$

with exact rows.

Applying $E\otimes -$, and using Case 2 on $K \hookrightarrow F$ and $\pi^{-1}(M) \hookrightarrow F$

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d, "\text{by Case2}"'] & \mathrm{Ker}(E\otimes j) \arrow[d] \arrow[dddll, bend left] & \\
 & E\otimes K \arrow[r] \arrow[d, "id_{E\otimes M}"] & E\otimes\pi^{-1}(M) \arrow[r] \arrow[d] & E\otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, "\text{by Case2}"'] & E\otimes K \arrow[r] \arrow[d] & E\otimes F \arrow[r] & E\otimes N \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

By the Snake Lemma $\mathrm{Ker}(E\otimes j) = 0$ $\square$